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A NOTE ON AN IDENTITY FOR THE DOUBLE SERIES WITH APPLICATIONS

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Abstract: Owing to the remarkable success of the hypergeometric function of one variable, the authors present a study of some family of hypergeometric functions of two variables (for example Kampé de Fériet's hypergeometric functions in two variables). The main aim of this paper is to provide several (presumably new) summation formulas for appropriately specified numerator and denominator parameters of the family in double hypergeometric functions having two equal arguments such as: $\frac{-1}{2}$, $\frac{1-\sqrt{2}}{2}$, $\frac{\pm 1}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{2-\sqrt{3}}{4}$, $\frac{-1}{8}$, $\frac{4-3\sqrt{2}}{8}$, $\frac{1}{9}$, $\frac{8}{9}$, $\frac{\sqrt{2}-1}{\sqrt{2}+1}$, $2\sqrt{2}-2$ and $12\sqrt{2}-16$. The methodology and techniques which are used in this paper, are based upon general double series identity obtained by using two-balanced summation theorem associated with Clausen hypergeometric polynomial with argument unity.

Keywords and Phrases: Kampé de Fériet's general double hypergeometric function; Cauchy's double series identity; Gamma function; Series rearrangement technique.

2020 Mathematics Subject Classification: 33C20, 33B99, 33C50, 40A25.

1. Introduction

In the usual notation, let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. Also let

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} \quad , \qquad \mathbb{N} = \{1, 2, 3, \dots\} = \mathbb{N}_0 \backslash \{0\} \quad ,$$

$$\mathbb{Z}_0^- = \{0, -1, -2, \dots\} = \mathbb{Z}^- \cup \{0\} \quad , \qquad \mathbb{Z}^- = \{-1, -2, -3, \dots\}$$

and $\mathbb{Z} = \mathbb{Z}_0^- \cup \mathbb{N}$ being the sets of integers.

1.1. Pochhammer symbol

In terms of Gamma function $\Gamma(z)$, the widely-used Pochhammer symbol $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined, in general, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}, \ \left(\lambda+\nu \quad and \quad \lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-\right)$$

$$= \begin{cases} 1, & \left(\nu = 0; \lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-\right), \\ \lambda(\lambda+1)\dots(\lambda+n-1) = \prod_{j=1}^{n}(\lambda+j-1), & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases} \tag{1.1}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ quotient exists see, for details, [43] and [44].

1.2. Generalized hypergeometric function

Here *generalized hypergeometric function* ${}_pF_q$ with p numerator parameters $\alpha_1, \alpha_2, \dots, \alpha_p$ and q denominator parameters $\beta_1, \beta_2, \dots, \beta_q$ is defined as, see ([35, pp. 71-72,] and [44, p. 41, et. seq.]).

$${}_pF_q\left[\begin{array}{c} \alpha_1, \alpha_2, \dots, \alpha_p; \\ \beta_1, \beta_2, \dots, \beta_q; \end{array} \ z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n} \frac{z^n}{n!}, \tag{1.2}$$

$$\Big(p,\ q \in \mathbb{N}_0;\ p \leqq q+1\ ;\ p \leqq q \text{ and } |z| < \infty\ ;\ p = q+1 \text{ and } |z| < 1;$$

$$p = q+1, |z| = 1 \text{ and } \Re(\omega) > 0;\ p = q+1, |z| = 1, z \neq 1 \text{ and } 0 \geq \Re(\omega) > -1\Big),$$

where

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j,$$

$$\left(\alpha_j \in \mathbb{C}\ (j = 1, 2, \ldots, p);\ \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-(j = 1, 2, \ldots, q)\right).$$

1.3. General double hypergeometric function of Kampé de Fériet

In 1921 Kampé de Fériet initiated the study of the case $B = C$ and $E = G$ of the double hypergeometric series:

$$F^{A:B;C}_{D:E;G}\left[\begin{array}{c} (a_A) : (b_B); (c_C); \\ (d_D) : (e_E); (g_G); \end{array} x, y\right] = \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{A}(a_j)_{m+n} \prod_{j=1}^{B}(b_j)_m \prod_{j=1}^{C}(c_j)_n}{\prod_{j=1}^{D}(d_j)_{m+n} \prod_{j=1}^{E}(e_j)_m \prod_{j=1}^{G}(g_j)_n} \frac{x^m}{m!} \frac{y^n}{n!}. \tag{1.3}$$

In the left hand side of equation (1.3), the notation for double hypergeometric function was given by Srivastava- panda [45, p.432, Equation (26)].

For the double hypergeometric series, Srivastava and Daoust [41] deduced from much more general results (proved by them) that

(i) If $A+B > D+E+1$ and $A+C > D+G+1$, then the series (1.3) diverges whenever $x \neq 0$ and $y \neq 0$.

(ii) If $A+B = D+E+1$ and $A+C = D+G+1$, then the series (1.3) converges absolutely, provided that

$$\max\{|x|, |y|\} < 1 \qquad \text{when} \quad A \leq D$$

$$|x|^{\frac{1}{A-D}} + |y|^{\frac{1}{A-D}} < 1 \qquad \text{when} \quad A > D.$$

(iii) If $A + B < D + E + 1$ and $A + C < D + G + 1$, then the series (1.3) converges absolutely for all $x, y \in \mathbb{C}$.

The Appell's double hypergeometric functions F_1, F_2, F_3 and F_4 [12, p. 224, Equations (5.7.1.6), ..., (5.7.1.9)] are denoted by $F^{1:1;1}_{1:0;0}$, $F^{1:1;1}_{0:1;1}$, $F^{0:2;2}_{1:0;0}$ and $F^{2:0;0}_{0:1;1}$ respectively and their seven confluent hypergeometric functions $\mathbf{\Phi}_1$, $\mathbf{\Phi}_2$, $\mathbf{\Phi}_3$, $\mathbf{\Psi}_1$, $\mathbf{\Psi}_2$, $\mathbf{\Xi}_1$, $\mathbf{\Xi}_2$ are given by Humbert [14], [15], see also [36] and [37].

For the absolutely and conditionally convergence of double hypergeometric series (1.3) readers can refer a stunning paper of Hài et-al. [13].

1.4. Cauchy's double series identity see [35] and [44]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k, n-k), \tag{1.4}$$

provided that both sides are absolutely convergent.

Remark 1.1. *Any values of parameters and arguments in following sections, leading to the results which do not make sense, are tacitly excluded.*

1.5. Two-balanced summation theorem

Prud III [26, p 539, Equation 7.4.4 (89)], Andrews-Askey-Roy [2, p. 179 Question 15]

$$
{}_3F_2\left[\begin{array}{c} -n, \quad a, \qquad\quad b \quad ; \\ c, 2+a+b-c-n; \end{array} 1\right] = \frac{(c-a-1)_n(c-b-1)_n}{(c)_n(c-a-b-1)_n}\frac{\left(\frac{ab+c^2-ac-bc-c}{c-a-b-1}\right)_n}{\left(\frac{ab+c^2-ac-bc-2c+a+b+1}{c-a-b-1}\right)_n}, \tag{1.5}
$$

$\left(n \in \mathbb{N}_0; a, b, c, 2+a+b-c-n,\ c-a-b-1, \frac{ab+c^2-ac-bc-2c+a+b+1}{c-a-b-1} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$.

Put $n = m, a = \beta, b = \alpha-\beta-\frac{1}{2}, c = \alpha+\beta+\frac{1}{2}$ in (1.5), we get

$$
{}_3F_2\left[\begin{array}{c} -m, \quad \beta, \quad \alpha-\beta-\frac{1}{2}\ ; \\ \alpha+\beta+\frac{1}{2}, 1-\beta-m; \end{array} 1\right] = \frac{\left(\alpha-\frac{1}{2}\right)_m(2\alpha)_m(2\beta)_m}{\left(\alpha+\beta+\frac{1}{2}\right)_m(\beta)_m(2\alpha-1)_m}, \tag{1.6}
$$

$$
\left(2\alpha-1,\ \beta\ , \alpha+\beta+\frac{1}{2}, 1-\beta-m \in \mathbb{C}\backslash\mathbb{Z}_0^- \ \text{ and } \ m \in \mathbb{N}_0\right).
$$

A number of expansion, reduction, summation and transformation formulas for hypergeometric function of two variables have been developed in the existing literature (see, for example, [9], [10], [11], [16], [41] and [42] and the references cited in each of these works.) Our present investigation is motivated by the work of [9], [10], [41], [42], [43], [44], [45] and [46] and other authors.

Further, in section 2 we obtained a general double series identity and a reduction formula for Kampé de Fériet double hypergeometric function in terms of generalized hypergeometric function of one variable.

In section 3, by specializing the numerator, denominator parameters and arguments in reduction formula, we obtain some interesting hypergeometric summations formulas for Kampé de Fériet double hypergeometric function having equal argument.

2. Main Results

In this section, we derived a double series identity involving bounded sequences, by the application of special case of two-balanced summation theorem for Clausen hypergeometric polynomial ${}_3F_2(1)$.

Theorem 2.1. *Let us assume that $\{\Phi(\mu)\}_{\mu=1}^{\infty}$ be a bounded sequence of essentially arbitrary complex or real numbers such that $\Phi(0) \neq 0$, then following general double* *series identity holds true:*

$$\sum_{m,n=0}^{\infty} \Phi(m+n) \frac{(\beta)_m(\beta)_n \left(\alpha-\beta-\frac{1}{2}\right)_n z^{m+n}}{\left(\alpha+\beta+\frac{1}{2}\right)_n m!n!} = \sum_{m=0}^{\infty} \Phi(m) \frac{(2\alpha)_m(2\beta)_m \left(\alpha-\frac{1}{2}\right)_m z^m}{\left(\alpha+\beta+\frac{1}{2}\right)_m (2\alpha-1)_m m!}, \tag{2.1}$$

$$\left(\alpha+\beta+\frac{1}{2},\quad 2\alpha-1 \in \mathbb{C}\backslash Z_0^-\ ;\qquad z\in\mathbb{C}\right),$$

provided that the infinite series occurring on both sides of the above assertions are absolutely convergent.

Proof of assertion (2.1).

Replacing m by $m-n$ in left hand side of assertion (2.1) and using Cauchy's double series identity (1.5), we get

$$\sum_{m,n=0}^{\infty} \Phi(m+n) \frac{(\beta)_m(\beta)_n \left(\alpha-\beta-\frac{1}{2}\right)_n z^{m+n}}{\left(\alpha+\beta+\frac{1}{2}\right)_n m!n!} =$$

$$= \sum_{m=0}^{\infty} \Phi(m) \frac{z^m(\beta)_m}{m!} \quad {}_3F_2\left[\begin{array}{r} -m, \beta, \alpha-\beta-\frac{1}{2}; \\ \alpha+\beta+\frac{1}{2}, 1-\beta-m; \end{array}\ 1\right]. \tag{2.2}$$

Now using the summation theorem (1.6), we get the right hand side of assertion (2.1).

2.1. Reducibility of the Kampé de Fériet double hypergeometric function

Now we establish a result for the reducibility of the Kampé de Fériet double hypergeometric function

$$\text{Put}\quad \Phi(\mu) = \frac{(d_1)_\mu (d_2)_\mu \ldots (d_D)_\mu}{(e_1)_\mu (e_2)_\mu \ldots (e_E)_\mu}\quad,\quad \mu\in\mathbb{N}_0.$$

in above double series identity (2.1) then applying the definition of double hypergeometric function of Kampé de Fériet and generalized hypergeometric function of one variable, we get following reduction formula

$$F^{D:2;1}_{E:1;0}\left[\begin{array}{l} (d_D) : \beta, \alpha-\beta-\frac{1}{2} : \beta; \\ (e_E) : \quad \alpha+\beta+\frac{1}{2}\quad ;-; \end{array}\ z, z\right] = {}_{D+3}F_{E+2}\left[\begin{array}{r} (d_D), 2\alpha, 2\beta, \alpha-\frac{1}{2}; \\ (e_E), \alpha+\beta+\frac{1}{2}, 2\alpha-1; \end{array}\ z\right], \tag{2.3}$$

$$\text{Where}\quad D=E,\quad |z|<1 \quad ;\qquad D\le E-1,\quad |z|<\infty.$$

A particular case of (2.3).

Put $D = 2$, $d_1 = \alpha + \beta + \frac{1}{2}$, $d_2 = 2\alpha - 1$; $E = 2$, $e_1 = \alpha - \frac{1}{2}$, $e_2 = g$ in main hypergeometric reduction formula (2.3), after simplification we get

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{r} \alpha+\beta+\frac{1}{2}, 2\alpha-1 : \beta,\ \alpha-\beta-\frac{1}{2};\ \ \beta; \\ \alpha-\frac{1}{2}, g :\ \ \ \alpha+\beta+\frac{1}{2}\ \ ;\ -; \end{array} z, z\right] = {}_2F_1\left[\begin{array}{r} 2\alpha, 2\beta; \\ g; \end{array} z\right]. \quad (2.4)$$

Both side of (2.4) are convergent when $|z| < 1$, for detail see convergence condition of equations (1.2) and (1.3).

3. Application

By specializing the numerator, denominator parameters and arguments in reduction formula (2.4), we shall obtain some summation theorem for double hypergeometric function of Kampé de Fériet as

- Put $\alpha = \frac{a}{4}, \beta = \frac{a+1}{4}, g = \frac{2a+2}{3}, z = \frac{8}{9}$ in (2.4) and applying the result recorded by Andrews et al. [2, p.131 Entry (3.1.20)], we get

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{l} \frac{2a+3}{4}, \frac{a-2}{2} : \frac{a+1}{4}, \frac{-3}{4}; \frac{a+1}{4}; \\ \\ \frac{a-2}{4}, \frac{2a+2}{3} :\ \ \frac{2a+3}{4}\ \ \ ;\text{---} ; \end{array} \frac{8}{9}, \frac{8}{9}\right] = \left(\frac{3}{2}\right)^a \frac{\sqrt{(\pi)}\Gamma(\frac{2a+2}{3})}{\Gamma(\frac{a+4}{6})\Gamma(\frac{a+1}{2})},$$

$$\left(\frac{2a+2}{3} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \quad (3.1)$$

- Put $\alpha = \frac{a}{4}, \beta = \frac{a+1}{4}, g = \frac{2a+5}{6}, z = \frac{1}{9}$ in (2.4) and applying the result recorded by Andrews et al. [2, p.131, Entry (3.1.17)]; Abramowitz [1, p.557, Entry(15.1.30)]; Kummer [18, v-4]; Per W. Karlsson [17, p.330, Equation (1.1)], we obtain

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{l} \frac{2a+3}{4}, \frac{a-2}{2} \ \ : \frac{a+1}{4}, \frac{-3}{4}; \frac{a+1}{4}; \\ \\ \frac{a-2}{4}, \frac{2a+5}{6} :\ \ \frac{2a+3}{4}\ \ ;\text{---} ; \end{array} \frac{1}{9}, \frac{1}{9}\right] =$$

$$= \left(\frac{3}{4}\right)^a \frac{\sqrt{(\pi)}\Gamma(\frac{2a+2}{3})}{\Gamma(\frac{a+4}{6})\Gamma(\frac{a+1}{2})} = \left(\frac{3}{4}\right)^{\frac{a}{2}} \frac{\sqrt{(\pi)}\Gamma(\frac{2a+5}{6})}{\Gamma(\frac{a+3}{6})\Gamma(\frac{a+5}{6})}\ ,\ \left(\frac{2a+5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \quad (3.2)$$

- Put $\alpha = \frac{a}{4}, \beta = \frac{2-a}{12}, g = \frac{2a+5}{6}, z = \frac{-1}{8}$ in (2.4) and applying the result recorded by Andrews et al. [2, p.177, Question 3(a)], we find

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{l} \frac{a+4}{6}, \frac{a-2}{2} \ \ : \frac{2-a}{12}, \frac{a-2}{3}; \frac{2-a}{12}; \\ \\ \frac{a-2}{4}, \frac{2a+5}{6} :\ \ \frac{a+4}{6}\ \ ;\text{---} ; \end{array} \frac{-1}{8}, \frac{-1}{8}\right] = (2)^{\frac{-a}{2}} \frac{\sqrt{(\pi)}\Gamma(\frac{2a+2}{3})}{\Gamma(\frac{a+4}{6})\Gamma(\frac{a+1}{2})},$$

$$\left(\frac{2a+5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.3}$$

- Put $\alpha = a, \beta = \frac{4a+1}{8}, g = \frac{4a+3}{4}, z = \frac{\sqrt{2}-1}{\sqrt{2}+1}$ in (2.4) and applying the result recorded by Andrews et al. [2, p.177, Question 3(b)], we have

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{c} \frac{12a+5}{8}, 2a-1 : \frac{4a+1}{8}, \frac{4a-5}{8}; \frac{4a+1}{8}; \\ \\ \frac{2a-1}{2}, \quad \frac{4a+3}{4} \quad : \quad \frac{12a+5}{8} \quad ; — \; ; \end{array} \frac{\sqrt{2}-1}{\sqrt{2}+1}, \frac{\sqrt{2}-1}{\sqrt{2}+1}\right] =$$

$$= (4-2\sqrt{2})^{-2a} \frac{\sqrt{(\pi)}\Gamma(\frac{4a+3}{4})}{\Gamma(\frac{2a+3}{4})\Gamma(\frac{a+1}{2})}, \qquad \left(a + \frac{3}{4} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.4}$$

- Put $\alpha = \frac{1}{4}, \beta = \frac{1-a}{2}, g = \frac{4a+1}{2}, z = \frac{1}{4}$ in (2.4) and applying the result recorded by Spiegel [40, p.894], Luke [23, p.273, Equation (6.8.20)] see also [21], [22] and [39], we acquire

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{c} \frac{5-2a}{4}, \frac{-1}{2} : \frac{1-a}{2}, \quad \frac{2a-3}{4}; \frac{1-a}{2}; \\ \\ \frac{-1}{4}, \frac{4a+1}{2} : \quad \frac{5-2a}{4} \quad ; — \; ; \end{array} \frac{1}{4}, \frac{1}{4}\right] = \frac{2\Gamma(a)\Gamma(\frac{4a+1}{2})}{3\Gamma(2a)\Gamma(\frac{2a+1}{2})},$$

$$\left(2a + \frac{1}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.5}$$

- Put $\alpha = \frac{a}{2}, \beta = \frac{2a+1}{4}, g = \frac{-4a+3}{2}, z = \frac{-1}{3}$ in (2.4) and applying the result recorded by Abramowitz [1, p.557, Entry (15.1.29)], Erd´elyi et al. [12, p.104, Entry (2.8.53)], see also Per W. Karlsson [17, p.334, Equation (2.12)], we achieve

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{c} \frac{4a+3}{4}, a-1 : \frac{2a+1}{4}, \quad \frac{1}{4}; \frac{2a+1}{4}; \\ \\ \frac{a-1}{2}, \frac{-4a+3}{2} : \quad \frac{4a+3}{4} \quad ; — \; ; \end{array} \frac{-1}{3}, \frac{-1}{3}\right] = \left(\frac{8}{9}\right)^{-2a} \frac{\Gamma(\frac{4}{3})\Gamma(\frac{-4a+3}{2})}{\Gamma(\frac{2}{3})\Gamma(\frac{-6a+4}{3})},$$

$$\left(-2a + \frac{3}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.6}$$

- Put $\alpha = \frac{a}{2}, \; \beta = \frac{3a+1}{6}, g = \frac{-3a+4}{3}, z = \frac{-1}{8}$ in (2.4) and applying the result recorded by Lavoie and Trottier [19, p.45, Equation (8)], we attain

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{c} \frac{3a+2}{3}, \; a-1 \; : \frac{3a+1}{6}, \quad \frac{-2}{3} \; ; \frac{3a+1}{6}; \\ \\ \frac{a-1}{2}, \frac{-3a+4}{3} \quad : \quad \frac{3a+2}{3} \quad ; — \; ; \end{array} \frac{-1}{8}, \frac{-1}{8}\right] = \left(\frac{2}{3}\right)^{3a} \frac{\Gamma(\frac{2-3a}{3})\Gamma(\frac{4-3a}{3})}{\Gamma(\frac{2}{3})\Gamma(\frac{4-6a}{3})},$$

$$\left(-a+\frac{4}{3}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.7}$$

- Put $\alpha=\frac{1-2a}{4}$, $\beta=a$, $g=\frac{6a+5}{6}$, $z=\frac{1}{9}$ in (2.4) and applying the result recorded by Per W. Karlsson [17, p.330, Equation (1.2)], we get

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{c} \frac{2a+3}{4},\frac{-2a-1}{2}:a\ ,\frac{-6a-1}{4};\ a\ ;\\ \\ \frac{-2a-1}{4},\frac{6a+5}{6}:\ \ \frac{2a+3}{4}\ \ ;—;\end{array}\ \frac{1}{9},\frac{1}{9}\right]=\frac{3^a\Gamma(\frac{6a+5}{6})\Gamma(\frac{2}{3})}{4^a\Gamma(\frac{3a+2}{3})\Gamma(\frac{5}{6})},$$

$$\left(a+\frac{5}{6}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.8}$$

- Put $\alpha=\frac{1-a}{2}$, $\beta=a$, $g=\frac{3a+2}{3}$, $z=\frac{1}{9}$ in (2.4) and applying the result recorded by Per W. Karlsson [17, p.330, Equation (1.3)], we obtain

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{c} \frac{2+a}{2},-a\ :\ a\ ,\frac{-3a}{2};\ \ a\ ;\\ \\ \frac{-a}{2},\frac{3a+2}{3}:\ \ \frac{2+a}{2}\ \ ;\ —;\end{array}\ \frac{1}{9},\frac{1}{9}\right]=\frac{3^a\sqrt{\pi}\Gamma(\frac{3a+2}{3})}{4^a\Gamma(\frac{2a+1}{2})\Gamma(\frac{2}{3})},$$

$$\left(a+\frac{2}{3}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.9}$$

- Put $\alpha=\frac{-a}{2}$, $\beta=\frac{2a+1}{2}$, $g=\frac{2}{3}$, $z=\frac{8}{9}$ in (2.4) and applying the result recorded by Per W. Karlsson [17, p.335, Equation (3.2)], we find

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{c} \frac{a+2}{2},-a-1:\frac{2a+1}{2}\ ,\frac{-3a-2}{2};\ \frac{2a+1}{2};\\ \\ \frac{-a-1}{2},\ \frac{2}{3}\ \ :\ \ \frac{a+2}{2}\ \ ;\ —;\end{array}\ \frac{8}{9},\frac{8}{9}\right]=2(3^a)\sin(\pi a+\frac{5\pi}{6}). \tag{3.10}$$

- Put $\alpha=\frac{-a}{2}$, $\beta=a+1$, $g=\frac{4}{3}$, $z=\frac{8}{9}$ in (2.4) and applying the result recorded by Per W. Karlsson [17, p.335, Equation (3.3)], we have

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{c} \frac{a+3}{2},-a-1:a+1\ ,\frac{-3a-3}{2};\ a+1\ ;\\ \\ \frac{-a-1}{2},\ \frac{4}{3}\ \ :\ \ \frac{a+3}{2}\ \ ;\ —;\end{array}\ \frac{8}{9},\frac{8}{9}\right]=\frac{(3^a)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{6}-a\right)\Gamma(a+\frac{3}{2})}. \tag{3.11}$$

- Put $\alpha = \frac{1-4a}{8}$, $\beta = \frac{-a}{2}$, $g = \frac{8a+5}{4}$, $z = \frac{1}{9}$ in (2.4) and applying the result recorded by Per W. Karlsson [17, p.330, Equation (1.4)], we acquire

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{lllll} \frac{5-8a}{8}, \frac{-4a-3}{4} & : \frac{-a}{2}, \frac{-3}{8} & ; \frac{-a}{2} & ; & \\ & & & & \frac{1}{9}, \frac{1}{9} \\ \frac{-4a-3}{8}, \frac{8a+5}{4} & : \frac{5-8a}{8} & ; \text{---} & ; & \end{array}\right] =$$

$$= \frac{(2^{6a})\Gamma\left(\frac{8a+5}{4}\right)\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{13}{12}\right)}{(3^{5a})\Gamma(\frac{3a+2}{3})\Gamma(\frac{12a+13}{12})\Gamma(\frac{5}{4}))} \quad , \left(2a + \frac{5}{4} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.12}$$

- Put $\alpha = \frac{1-4a}{8}$, $\beta = \frac{-a}{2}$, $g = \frac{8a+9}{4}$, $z = \frac{1}{9}$ in (2.4) and applying the result recorded by Per W. Karlsson [17, p.330, Equation (1.5)], we achieve

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{lllll} \frac{5-8a}{8}, \frac{-4a-3}{4} & : \frac{-a}{2}, \frac{-3}{8} & ; \frac{-a}{2} & ; & \\ & & & & \frac{1}{9}, \frac{1}{9} \\ \frac{-4a-3}{8}, \frac{8a+9}{4} & : \frac{5-8a}{8} & ; \text{---} & ; & \end{array}\right] =$$

$$= \frac{(2^{6a})\Gamma\left(\frac{8a+9}{4}\right)\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{17}{12}\right)}{(3^{5a})\Gamma(\frac{3a+4}{3})\Gamma\left(\frac{12a+17}{12}\right)\Gamma\left(\frac{9}{4}\right)} \quad , \left(2a + \frac{9}{4} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.13}$$

- Put $\alpha = \frac{3a}{2}$, $\beta = \frac{12a+1}{8}$, $g = \frac{12a+1}{3}$, $z = \frac{8}{9}$ in (2.4) and applying the result recorded by Per W. Karlsson [17, p.335, Equation (3.4)], we attain

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{lllll} \frac{24a+5}{8}, 3a-1 & : \frac{12a+1}{8}, \frac{-5}{8} & ; \frac{12a+1}{8} & ; & \\ & & & & \frac{8}{9}, \frac{8}{9} \\ \frac{3a-1}{2}, \frac{12a+1}{3} & : \frac{24a+5}{8} & ; \text{---} & ; & \end{array}\right] =$$

$$= \frac{(108)^a\Gamma\left(\frac{12a+7}{12}\right)\Gamma\left(\frac{6a+5}{6}\right)\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{4a+3}{4}\right)\Gamma(\frac{3a+2}{3})\Gamma(\frac{7}{12})\Gamma\left(\frac{5}{6}\right)} \quad , \left(4a + \frac{1}{3} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.14}$$

- Put $\alpha = \frac{3a}{2}$, $\beta = \frac{12a-1}{8}$, $g = \frac{12a+1}{3}$, $z = \frac{8}{9}$ in (2.4) and applying the result recorded by Per W. Karlsson [17, p.335, Equation (3.5)], we get

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{lllll} \frac{24a+3}{8}, 3a-1 & : \frac{12a-1}{8}, \frac{-3}{8} & ; \frac{12a-1}{8} & ; & \\ & & & & \frac{8}{9}, \frac{8}{9} \\ \frac{3a-1}{2}, \frac{12a+1}{3} & : \frac{24a+3}{8} & ; \text{---} & ; & \end{array}\right] =$$

$$= \frac{(108)^a\Gamma\left(\frac{12a+1}{12}\right)\Gamma\left(\frac{6a+5}{6}\right)\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma(\frac{4a+1}{4})\Gamma(\frac{3a+2}{3})\Gamma(\frac{1}{12})\Gamma\left(\frac{5}{6}\right)} \quad , \left(4a + \frac{1}{3} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.15}$$

- Put $\alpha = \frac{1+6a}{4}$, $\beta = \frac{3a}{2}$, $g = \frac{12a+2}{3}$, $z = \frac{8}{9}$ in (2.4) and applying the result recorded by Kummer [18, (v-6)]Kummer 1836, see also Per W. Karlsson [17, p.335, Equation (3.1)], we obtain

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{ccccc} \frac{12a+3}{4}, \frac{6a-1}{2} & : \frac{3a}{2}, \frac{-1}{4}; & \frac{3a}{2}; & \\ & & & \frac{8}{9}, \frac{8}{9} \\ \frac{6a-1}{4}, \frac{12a+2}{3} & : \frac{12a+3}{4}; & —; & \end{array}\right] = \frac{(27)^a\Gamma\left(\frac{12a+5}{6}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\frac{6a+5}{6})\Gamma(\frac{2a+1}{2})},$$

$$\left(4a + \frac{2}{3} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.16}$$

- Put $\alpha = \frac{a}{2}$, $\beta = \frac{1}{4}$, $g = 3a$, $z = \frac{3}{4}$ in (2.4) and applying the result recorded by Luke [23, p.273, Equation (6.8.18)], we find

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{ccccc} \frac{2a+3}{4}, a-1 : & \frac{1}{4}, \frac{2a-3}{4}; & \frac{1}{4}; & \\ & & & \frac{3}{4}, \frac{3}{4} \\ \frac{a-1}{2}, 3a : & \frac{2a+3}{4}; & —; & \end{array}\right] = \left(\frac{16}{27}\right)^a \frac{\Gamma(a)\Gamma(3a)}{[\Gamma(2a)]^2},$$

$$\left(3a \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.17}$$

- Put $\alpha = 1$, $\beta = \frac{a}{2}$, $g = \frac{5-a}{2}$, $z = \frac{-1}{2}$ in (2.4) and applying the result recorded by Prudnikov et al. [26, p.494, Equation (13)], we have

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{ccccc} \frac{a+3}{2}, 1 : & \frac{a}{2}, \frac{1-a}{2}; & \frac{a}{2}; & \\ & & & \frac{-1}{2}, \frac{-1}{2} \\ \frac{1}{2}, \frac{5-a}{2} : & \frac{a+3}{2}; & —; & \end{array}\right] = \frac{3-a}{3}, \quad \left(\frac{5-a}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.18}$$

- Put $\alpha = \frac{a}{2}$, $\beta = \frac{2-a}{4}$, $g = \frac{2a+1}{2}$, $z = \frac{-1}{3}$ in (2.4) and applying the result recorded by Prudnikov et al. [26, p.494, Equation (14)], we acquire

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{ccccc} \frac{a+4}{4}, a-1 : & \frac{2-a}{4}, \frac{3a-4}{4}; & \frac{2-a}{4}; & \\ & & & \frac{-1}{3}, \frac{-1}{3} \\ \frac{a-1}{2}, \frac{2a+1}{2} : & \frac{a+4}{4}; & —; & \end{array}\right] = \frac{\sqrt{\pi}\ \Gamma(\frac{2a+1}{2})}{3^{(\frac{a}{2})}\left[\Gamma\left(\frac{a+1}{2}\right)\right]^2},$$

$$\left(\frac{2a+1}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.19}$$

- Put $\alpha = \frac{1}{4}$, $\beta = \frac{a}{2}$, $g = \frac{2a+3}{4}$, $z = \frac{1-\sqrt{2}}{2}$ in (2.4) and applying the result recorded by Prudnikov et al. [26, p.494, Equation (16)], we achieve

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{ccccc} \frac{2a+3}{4}, \frac{-1}{2} : & \frac{a}{2}, \frac{-2a-1}{4}; & \frac{a}{2}; & \\ & & & \frac{1-\sqrt{2}}{2}, \frac{1-\sqrt{2}}{2} \\ \frac{-1}{4}, \frac{2a+3}{4} : & \frac{2a+3}{4}; & —; & \end{array}\right]$$

$$= 2^{\frac{-a}{2}}\sqrt{\pi}\frac{\Gamma(\frac{2a+3}{4})}{\Gamma(\frac{a+2}{4})\Gamma(\frac{a+3}{4})}, \qquad \left(\frac{2a+3}{4} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.20}$$

- Put $\alpha = \frac{a}{2}$, $\beta = \frac{2-a}{6}$, $g = \frac{2a+5}{6}$, $z = \frac{4-3\sqrt{2}}{8}$ in (2.4) and applying the result recorded by Prudnikov et al. [26, 98, p.495, Equation (22)], we attain

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{llll} \frac{2a+5}{6}, a-1 & : \frac{2-a}{6}, \frac{4a-5}{6} & ; \frac{2-a}{6}; & \frac{4-3\sqrt{2}}{8}, \frac{4-3\sqrt{2}}{8} \\ \frac{a-1}{2}, \frac{2a+5}{6} & : \frac{2a+5}{6} & ; \text{—}; & \end{array}\right]$$

$$= \left(\frac{2}{3}\right)^{\frac{a}{2}}(\sqrt{\pi})\frac{\Gamma(\frac{2a+5}{6})}{\Gamma(\frac{a+3}{6})\Gamma(\frac{a+5}{6})}, \qquad \left(\frac{2a+5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.21}$$

- Put $\alpha = \frac{a}{2}$, $\beta = \frac{2-3a}{2}$, $g = \frac{3-2a}{2}$, $z = \frac{2-\sqrt{3}}{4}$ in (2.4) and applying the result recorded by Prudnikov et al. [26, 98, p.495, Equation (25)], we get

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{llll} \frac{3-2a}{2}, a-1 & : \frac{2-3a}{2}, \frac{4a-3}{2}; & \frac{2-3a}{2}; & \frac{2-\sqrt{3}}{4}, \frac{2-\sqrt{3}}{4} \\ \frac{a-1}{2}, \frac{3-2a}{2} & : \frac{3-2a}{2} & ; \text{—}; & \end{array}\right]$$

$$= \frac{3^{\frac{3a}{2}}\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{3-2a}{2}\right)}{2^{(2a-1)}\sqrt{\pi}\,\Gamma\left(\frac{4-3a}{3}\right)}, \qquad \left(\frac{3-2a}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.22}$$

- Put $\alpha = \frac{a}{2}$, $\beta = \frac{1-2a}{2}$, $g = \frac{4-3a}{3}$, $z = \frac{1}{9}$ in (2.4) and applying the result recorded by Prudnikov et al. [26, 98, p.495, Equation (27)], we obtain

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{llll} \frac{2-a}{2}, a-1 & : \frac{1-2a}{2}, \frac{3a-2}{2}; & \frac{1-2a}{2}; & \frac{1}{9}, \frac{1}{9} \\ \frac{a-1}{2}, \frac{4-3a}{3} & : \frac{2-a}{2} & ; \text{—}; & \end{array}\right] = \frac{3^{-a}\Gamma\left(\frac{2-3a}{3}\right)\Gamma\left(\frac{4-3a}{3}\right)}{\Gamma\left(\frac{2}{3}\right)\ \Gamma\left(\frac{4-6a}{3}\right)},$$

$$\left(\frac{4-3a}{3} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.23}$$

- Put $\alpha = \frac{1}{4}$, $\beta = \frac{a}{2}$, $g = \frac{5-4a}{2}$, $z = \frac{1}{4}$ in (2.4) and applying the result recorded by Prudnikov et al. [26, 98, p.495, Equation (31)], we find

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{llll} \frac{2a+3}{4}, \frac{-1}{2} & : \frac{a}{2}, \frac{-2a-1}{4}; & \frac{a}{2}; & \frac{1}{4}, \frac{1}{4} \\ \frac{-1}{4}, \frac{5-4a}{2} & : \frac{2a+3}{4} & ; \text{—}; & \end{array}\right] = \frac{2}{3}^{2a}\sqrt{(\pi)}\frac{\Gamma\left(\frac{5-4a}{2}\right)}{[\Gamma\left(\frac{3-2a}{2}\right)]^2},$$

$$\left(\frac{5-4a}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.24}$$

- Put $\alpha = \frac{a}{2}$, $\beta = \frac{1-2a}{2}$, $g = a+2$, $z = \frac{1}{3}$ in (2.4) and applying the result recorded by Prudnikov et al. [26, 98, p.495, Equation (34)], we have

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{lllll} \frac{2-a}{2}, a-1 & : \frac{1-2a}{2} & , \frac{3a-2}{2}; & \frac{1-2a}{2}; & \\ & & & & \frac{1}{3}, \frac{1}{3} \\ \frac{a-1}{2}, \ a+2 & : \quad \frac{2-a}{2} & ; & \text{—} ; & \end{array}\right] = \left(\frac{2}{3}\right)^{2a}(a+1),$$

$$\left(a+2 \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.25}$$

- Put $\alpha = \frac{a}{2}$, $\beta = \frac{2a+1}{8}$, $g = \frac{2a+1}{2}$, $z = 2\sqrt{2}-2$ in (2.4) and applying the result recorded by Prudnikov et al. [26, 98, p.495, Equation (36)], we acquire

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{lllll} \frac{6a+5}{8}, a-1 : & \frac{2a+1}{8} & , \frac{2a-5}{8}; & \frac{2a+1}{8}; & \\ & & & & (2\sqrt{2}-2), \ (2\sqrt{2}-2) \\ \frac{a-1}{2}, \ \frac{2a+1}{2} : & \frac{6a+5}{8} & ; & \text{—} ; & \end{array}\right] =$$

$$= \left(\frac{2+\sqrt{2}}{2}\right)^{a} \sqrt{\pi}\frac{\Gamma\left(\frac{2a+3}{4}\right)}{\Gamma\left(\frac{a+2}{4}\right)\Gamma\left(\frac{a+3}{4}\right)}, \ \left(\frac{2a+1}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.26}$$

- Put $\alpha = \frac{a}{2}$, $\beta = \frac{2a+1}{4}$, $g = 4a$, $z = \frac{8}{9}$ in (2.4) and applying the result recorded by Prudnikov et al. [26, p.496, Equation (41)], we achieve

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{lllll} \frac{4a+3}{4}, a-1 : & \frac{2a+1}{4} & , \frac{-3}{4}; & \frac{2a+1}{4}; & \\ & & & & \frac{8}{9}, \frac{8}{9} \\ \frac{a-1}{2}, \ 4a : & \frac{4a+3}{4} & ; & \text{—} ; & \end{array}\right] = 2^{1-6a} \times 3^{2a}\sqrt{\pi}\frac{\Gamma(4a)}{\Gamma\left(\frac{2a+1}{2}\right)\Gamma(3a)},$$

$$\left(4a \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{3.27}$$

- Put $\alpha = \frac{a}{2}$, $\beta = \frac{4a+1}{12}$, $g = \frac{4a+1}{3}$, $z = 12\sqrt{2}-16$ in (2.4) and applying the result recorded by Prudnikov et al. [26, p.496, Equation (43)], see also Brychkov [5, p.587, Equation (172)], we attain

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{lllll} \frac{10a+7}{12}, a-1 : & \frac{4a+1}{12} & , \frac{2a-7}{12}; & \frac{4a+1}{12}; & \\ & & & & 12\sqrt{2}-16, 12\sqrt{2}-16 \\ \frac{a-1}{2}, \ \frac{4a+1}{3} : & \frac{10a+7}{12} & ; & \text{—} ; & \end{array}\right]$$

$$= \left(\frac{2+\sqrt{2}}{2}\right)^{2a} \sqrt{\pi}\frac{\Gamma\left(\frac{2a+2}{3}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{a+4}{6}\right)} = \frac{3^{\frac{-a}{2}}(3+2\sqrt{2})^{a}\sqrt{\pi}\ \Gamma\left(\frac{2a+5}{6}\right)}{\Gamma\left(\frac{a+3}{6}\right)\Gamma\left(\frac{a+5}{6}\right)}, \tag{3.28}$$

$$\left(\frac{4a+1}{3} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

- Put $\alpha = \frac{a}{2},\ \beta = \frac{4a-1}{4},\ g = 4a-1,\ z = 12\sqrt{2} - 16$ in (2.4) and applying the result recorded by Prudnikov et al. [26, p.496, Equation (46)], we get

$$F^{2:2;1}_{2:1;0}\left[\begin{array}{ccc} \frac{6a+1}{4},\ a-1 & : \frac{4a-1}{4}, \frac{-2a-1}{4}; & \frac{4a-1}{4}; \\ & & \\ \frac{a-1}{2},\ 4a-1 & : \quad \frac{6a+1}{4} \quad ; & \text{---}\ ; \end{array} (12\sqrt{2}-16), (12\sqrt{2}-16)\right]$$

$$= \frac{(2+\sqrt{2})^{2a}}{2^{(4a-1)}}\sqrt{\pi}\frac{\Gamma(2a)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{3a}{2}\right)}\ ,\ \left(4a-1 \in \mathbb{C}\backslash\mathbb{Z}_0^-\ \right). \tag{3.29}$$

4. Conclusion

In our investigation herein, we have presented several potentially useful summation theorems involving a family of double hypergeometric functions. In conclusion we observe that many similar types of general double series identities which are related to other mathematical functions and are different from the given double series identity, can also be evaluated and applied in an analogous manner. We remark further that many of the results, which we have derived in this paper, are quite significant and are expected to be beneficial for the researchers in several fields of applied mathematics and mathematical sciences, and also in other branches of science and engineering.

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References

[1] Abramowitz, M. and Stegun, I. A., Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards Applied Mathematics Series, Vol. 55, U. S. Government Printing Office, Washington, 1964.

[2] Andrews, G. E., Askey, R. and Roy, R., Special Functions, Encyclopedia of Mathematics and its Applications, Vol. 71, Cambridge University Press, Cambridge, 1999.

[3] Bailey, W. N., Generalized Hypergeometric Series, Cambridge Math. Tract No. 32, Cambridge University Press, Cambridge, 1935; Reprinted by Stechert-Hafner, New York, 1964.

[4] Bhawna, G. Qureshi, M. I. and Baboo, M. S., A note on Nielsen-type integrals, Logarithmic integrals and higher Harmonic sums, J. Computational Analysis and Applications, 33(1) (2024), 33-47.

[5] Brychkov, Yury A., Handbook of Special Functions : Derivatives, Integrals, Series and Other Formulas, A Chapman & Hall book, CRC Press (Taylor and Francis Group), Boca Raton, London, New York, 2008.

[6] Chaudhary, W. M., Ara, J. and Qureshi, M. I., On certain linear, Bilinear and Bilateral hypergeometric generating relations, Int. J. Appl. Comput. Math., (2023), 1-11.

[7] Chen, K.-Y., Liu, S.-J. and Srivastava, H. M., Some double-series identities ans associated generating-function relationships, Applied Mathematics Letters, 19 (2006), 887-893.

[8] Chen, K.-Y. and Srivastava, H. M., Series Identities and Associated Families of Generating Functions, J. Math. Anal. Appl., 311(2) (2005), 582-599.

[9] Choi, J. and Rathie, A. K., General summation formula for the Kampé de Fériet function, Montes Taurus J. Pure. Appl. Math., I (2019), 107-128.

[10] Choi, J. and Srivastava, H. M., A Reducible Case of Double Hypergeometric Series Involving the Riemann ζ- Function, Bull. Korean Math. Soc., 33(1) (1996), 107-110.

[11] Cvijoviȁ, D. and Miller, A. R., A reduction formula for the Kampé de Fériet function, Appl. Math. Lett., 23 (2010), 769-771.

[12] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G., Higher Transcendental Functions, Vol. I, McGraw -Hill Book Company, New York, Toronto and London, 1953.

[13] H$\grave{a}$i, N. T., Marichev, O. I. and Srivastava, H. M., A note on the convergence of certain families of multiple hypergeometric series, J. Math. Anal. Appl., 164 (1992), 104-115.

[14] Humbert, P., La fonction $W_{k,\mu_1,\mu_2,\ldots,\mu_n}(x_1, x_2, \ldots, x_n)$, C. R. Acad. Sci. Paris, 171 (1920), 428-430.

[15] Humbert, P., The confluent hypergeometric functions of two variables, Proc. Royal Soc. Edinburgh Sect. A., 41 (1920-21), 73-96.

[16] Karlsson, Per W., Reduction of certain generalized Kampé de Fériet Functions, Math. Scand., 32 (1973), 265-268.

[17] Karlsson, Per W., On two Hypergeometric Summation Formulas Conjectured by Gosper, Simon Stevin, 60(4) (1986), 329-337.

[18] Kummer, E. E., Über die hypergeometrische Reihe

$$1+\frac{\alpha.\beta}{1.\gamma}x+\frac{\alpha(\alpha+1).\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2).\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3+\ldots,$$

J. Reine Angew. Math. 15 (1836), 39-83 and 127-172; see also Collected papers, Vol. II: Function Theory, Geometry and Miscellaneous (Edited and with a Foreword by André Weil), Springer-Verlag, Berlin, Heidelberg and New York, 1975.

[19] Lavoie, J. L. and Trottier, G., On the Sum of Certain Appell Series, Ganita, 20(1) (1969), 43-46.

[20] Lin, H. M. and Wang, W. P., Transformation and summation formulae for Kampé de Fériet series, J. Math. Anal. Appl., 409 (2014), 100-110.

[21] Luke, Y. L., The Special Functions and Their Approximations, Vol.I, Academic Press, London, 1969.

[22] Luke, Y. L., The Special Functions and Their Approximations, Vol.II, Academic Press, London, 1969.

[23] Luke, Y. L., Mathematical Functions and Their Approximations, Academic Press, London, 1975.

[24] Pathan, M. A., Qureshi, M. I. and Majid, J., Analytical expression for the exact curved surface area and volume of hyperboloid of two sheets via Mellin-Barnes type contour integration, Journal of Mahani Mathematical Research, 12(2) (2023), 77-104.

[25] Pathan, M. A., Qureshi, M. I., Malik, S. H. and Bhat, B. A., A note on Karlsson-Srivastava terminating series ${}_3F_2(4)$ and ${}_3F_2(\frac{1}{4})$ with applications, Montes Taurus J. Pure. Appl. Math., 5(2) (2023), 51-56.

[26] Prudnikov, A. P., Brychkov, Yu. A. and Marichev, O. I., Integrals and Series, Volume III : More Special Functions, Nauka, Moscow, 1986 (In Russian); (Translated from the Russian by G.G.Gould) Gordon and Breach Science Publishers, New York, 1990.

[27] Qureshi, M. I., Bhat, A. H. and Majid, J., Hypergeometric form of $(1+x^2)^{\frac{ib}{2}} exp(b\tan^{-1} x)$ and its applications, Jñānābha, 53(1) (2023), 87-91.

[28] Qureshi, M. I. and Bhat, B. A., Addition theorems for the product of two generalized Hermite polynomials and their consequences, Punjab University Journal of Mathematics, 56(3/4) (2024), 112-122.

[29] Qureshi, M. I., Chaudhary, W. M. and Khan, M. K., Some Transformations for Goursat's Function ${}_2F_2[2x]$, South East Asian J. of Mathematics and Mathematical Sciences, 18(1) (2022), 179-190.

[30] Qureshi, M. I., Khammash, G. S., Bhat, A. H. and Majid, J., Some extensions and generalizations of Kümmer's third summation theorem, Al-Mukhtar Journal of Sciences, 37(4) (2022), 329-344.

[31] Qureshi, M. I. and Malik, S. H., Evaluation of certain of families of log-cosine integrals using hypergeometric function approach and applications, Notes on Number Theory and Discrete Mathematics, 30(3) (2024), 499-515.

[32] Qureshi, M. I., Malik, S. H. and Ara, J., Hypergeometric Reduction Formulas Involving the Gauss and Clausen Functions, GANITA, 71(1) (2021), 269-273.

[33] Qureshi, M. I., Quraishi, K. A., Chaudhary, W. M. And Khan, M. K., Double Series Identity, Gaussian Quadratic Transformation and its Consequences, J. Comput. Math. Sci., 9(12) (2018), 2127-2137.

[34] Qureshi, M. I., Shah, T. R., Choi, J. and Bhat, A. H., Three general Double-series identities and associated reduction formulas and fractional calculus, Fractal and Fractional, MDPI, 7 (2023), 1-26.

[35] Rainville, E. D., Special Functions, The Macmillan Company, New York, 1960 ; Reprinted by Chelsea Publ. Co., Bronx, New York, 1971.

[36] Saigo, M., On properties of the Lauricella hypergeometric function F_D, Bull. Central Res. Inst. Fukuoka Univ., 104 (1988), 13-31.

[37] Saigo, M., On properties of hypergeometric functions of three variables, F_M and F_G, Rend. Circ. Mat. Palermo, Series II, 37 (1988), 449-468.

[38] Slater, L. J., Generalized Hypergeometric Functions, Cambridge University Press, Cambridge, 1966.

[39] Spiegel, M. R., Some Interesting Cases of the Hypergeometric Series, Amer. Math. Monthly, 69 (1962), 894-896.

[40] Spiegel, M. R., Mathematical Handbook of Formulas and Tables, McGraw-Hill Book Company, New York, 1968.

[41] Srivastava, H. M. and Daoust, M. C., A note on the convergence of Kampé de Fériet's double hypergeometric series, Mad Nachr., 53 (1972), 151-159.

[42] Srivastava, H. M., Gupta, B., Qureshi, M. I. and Baboo, M. S., Some summation theorems and transformations for hypergeometric functions of Kampé de Fériet and Srivastava, Georgian Math. J. (De Gruyter), (2024), 1-13.

[43] Srivastava, H. M. and Karlsson, P. W., Multiple Gaussian Hypergeometric Series, Halsted Press (Ellis Horwood Limited, Chichester, U.K.), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

[44] Srivastava, H. M. and Manocha, H. L., A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[45] Srivastava, H. M. and Panda, R., An Integral Representation for the Product of two Jacobi Polynomials, J. London Math. Soc., 2(12) (1976), 419-425.

[46] Srivastava, H. M. and Pathan, M. A., Some bilateral generating functions for the extended Jacobi polynomials-I, Comment. Math. Univ. St. Paul., 28(1) (1979), 23-30.

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